Linear Algebra II

01/03/2021, Monday, 18:30 – 20:30

1 $(5 + 10 + 10 = 25 \text{ pts})$ **Least squares approximation**

For a given matrix $A \in \mathbb{R}^{m \times n}$ with with rank n, let $A = QR$ be a QR-decomposition.

(a) Prove that $A^\top A$ is nonsingular and that $(A^\top A)^{-1} = R^{-1}(R^{-1})^\top$.

Now consider the matrix

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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(b) Determine a QR-decomposition of A.

(c) Consider the linear equation

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Determine the least squares solution to this equation.

2 $(6 + 6 + 5 + 8 = 25 \text{ pts})$ **Inner product spaces**

Consider the real vector space $C[0,1]$ of continuous functions on $[0,1]$

(a) For $f, g \in C[0,1]$, define $\langle f, g \rangle := \int_0^1 f(x)g(x)dx + 2f(0)g(0) + f(1)g(1)$. Prove that this defines an inner product on $C[0,1]$.

Let S be the subspace spanned by the functions $f(x) = 1$ and $g(x) = x$.

(b) Compute $\|f\|$ and $\|g\|$.

(c) Let θ be the angle between f and g. Compute $\cos\theta$.

(d) Find an orthonormal basis of S.

3 (5 + 5 + 5 + 5 = 20 pts) **Eigenvalues**

Let A be a the matrix given by

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Determine the eigenvalues of A.

(c) Determine corresponding eigenvectors.

(d) Is A diagonalizable? Explain your answer.

4 (5 + 5 + 5 + 5 = 20 pts) **Diagonalization**

Let $A \in \mathbb{C}^{n\times n}$ be a unitary matrix.

(a) Prove that every eigenvalue λ of A lies on the unit circle, i.e. $|\lambda| = 1$.

(b) Prove that if a complex number λ satisfies $|\lambda| = 1$, then $\overline{\lambda} = \frac{1}{\lambda}$.

(c) Show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(d) Is A is unitarily diagonalizable? Explain your answer.

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10 pts free